Prescribed Scalar Curvature with Minimal Boundary Mean Curvature on S_+^4

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Abstract. This paper is devoted to the prescribed scalar curvature under minimal boundary mean curvature on the standard four dimensional half sphere. Using topological methods from the theory of critical points at infinity, we prove some existence results.

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1 Introduction

In this paper, we are interested in some nonlinear equation arising from a geometric context. Namely, let

$$S_+^n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \, / \, |x| = 1, \, x_{n+1} > 0\}$$

be the standard half sphere endowed with its standard metric g, $n \geq 3$ and let
$f : S_+^n \to \mathbb{R}$ be a given function and we consider the following problem : does there exist a metric $\tilde{g}$ conformally equivalent to g such that $R_{\tilde{g}} \equiv f$ and $h_{\tilde{g}} \equiv 0$? where $R_{\tilde{g}}$ is the scalar curvature of S_+^n and $h_{\tilde{g}}$ is the mean curvature of ∂S_+^n, with respect to $\tilde{g}$. Setting $\tilde{g} = u^{4/(n-2)}g$ a conformal metric to g, where u is a smooth positive function, the above problem has the following analytical formulation : find a smooth positive function which solves the following problem

$$(P) \quad \begin{cases} L_g u := -\Delta_g u + \frac{n(n-2)}{4}u = K u^{\frac{n+2}{n-2}} & \text{in} \quad S_+^n \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial S_+^n \end{cases}$$

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where ν is the outer normal vector with respect to g and where $K = \frac{n-2}{4(n-1)}f$.

Such kind of problem (P) has attracted much attention (see [1] [10], [12], [13], [16],[18] [19], [20], [21], [22] [24] and the references therein).

The main difficulty one encounters in problem (P) appears when we consider it from a variational viewpoint, indeed, the Euler functional associated to (P) does not satisfy the Palais-Smale condition, that is, there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. This fact is due to the presence of the critical Sobolev exponent in (P). Moreover, there are topological obstructions to solving (P), based on the Kazdan-Warner type condition, see [14]. Hence it is not expectable to solve problem (P) for all functions K, and so a natural question arises : under which conditions on K, (P) has a positive solution?

Yanyan Li [24] and Djadli, Malchiodi and Ould Ahmedou [18] studied problem (P) on the three dimensional standard half sphere. Their method involves a fine blow up analysis of some subcritical approximations and the use of the topological degree tools. Ben Ayed, El Mehdi and Ould Ahmedou [12], [13] gave some sufficient topological conditions on K to find solutions to (P) for n bigger than or equal to 4. Their approach uses algebraic topological tools from the theory of critical points at infinity (see Bahri [4]).

Notice that problem (P) is, in some sense, related to the well-known Scalar Curvature problem on S^n

$$(P') \qquad -\Delta_g u + \frac{n(n-2)}{4}u = K u^{(n+2)/(n-2)} \qquad \text{in } S^n$$

to which many works have been devoted (see for example the monographes [2], [23] and references therein.)

Regarding problem (P'), there is a difference between the three cases $n = 3$, $n = 4$ and higher dimensions. In the case $n = 3$, the interaction between two of the functions failing the Palais-Smale condition dominates the self interaction, while for $n = 4$, there is a balance phenomenon, and for $n \geq 5$ the self interaction dominates the interaction of two of those functions (see [5], [7], [9]).

For problem (P), such a balance phenomenon (i.e. the self interaction and the interaction are of the same size) appears in dimensions 3 and 4, see [18], [13]. In this work, we focus on the four dimensional case to give more existence results. We are thus reduced to find positive solutions of the following problem

$$(1) \qquad \begin{cases} -\Delta_g u + 2u & = & K u^3 & \text{in} & S_+^4 \\ \frac{\partial u}{\partial \nu} & = & 0 & \text{on} & \partial S_+^4 \end{cases}$$

Precisely, we borrow some of the ideas developed in Bahri [5], Aubin-Bahri [3], Ben Ayed-Chtioui-Hammami [11], Ben Ayed-El Mehdi-Ould Ahmedou [12], [13] and Chtioui [17]. The main idea is to precise the topological contribution of the critical points at infinity between the level sets of the associated Euler functional and the main issue is under our conditions on K, there remains some difference of topology which is not due to the critical

points at infinity and therefore the existence of a solution of (1).

In order to state our results, we fix some notations and assumptions that we are using in our results.

Let G be the Green's function of L_g on S_+^4 and H its regular part defined by

$$\begin{cases} G(x,y) = (1 - \cos(d(x,y)))^{-1} - H(x,y), \\ \Delta H = 0 \text{ in } S_+^4, \quad \partial G/\partial \nu = 0 \text{ on } \partial S_+^4 \end{cases}$$

Let K be a C^3 positive function on $\overline{S_+^4}$.

Throughout this paper, we assume that the following two assumptions hold

(1.1) K has only nondegenerate critical points $y_0, y_1, ..., y_m$ such that y_0 is the unique absolute maximum of K on $\overline{S_+^4}$ and such that

$$\frac{-\Delta K(y_i)}{3K(y_i)} + 4H(y_i, y_i) \neq 0, \qquad \text{for } i = 0, 1, ..., m$$

(1.2)   All the critical points of $K_1 = K_{/\partial S_+^4}$ are $z_1, ..., z_{m'}$, and satisfy

$$\frac{\partial K}{\partial \nu}(z_i) < 0, \qquad \text{for } i = 1, ..., m'$$

Now we introduce the following set

$$\mathcal{F}^+ = \{y \in S_+^4 / \nabla K(y) = 0 \text{ and } \frac{-\Delta K(y_i)}{3K(y_i)} + 4H(y_i, y_i) > 0\}$$

Thus we are able to state our first result

Theorem 1.1 *If $y_0 \notin \mathcal{F}^+$, then problem (1) has a solution.*

In the above result, we have assumed that $y_0 \notin \mathcal{F}^+$. Next we want to give some existence result for problem (1) when $y_0 \in \mathcal{F}^+$. To this aim, we introduce some notation.

For $s \in \mathbb{N}^*$ and for any s-tuple $\tau_s = (i_1, ..., i_s) \in (\mathcal{F}^+)^s$ such that $i_p \neq i_q$ if $p \neq q$, we define a Matrix $M(\tau_s) = (M_{pq})_{1 \leq p,q \leq s}$, by

$$M_{pp} = \frac{-\Delta K(y_{i_p})}{3K(y_{i_p})^2} + 4\frac{H(y_{i_p}, y_{i_p})}{K(y_{i_p})}, \qquad M_{pq} = -\frac{4G(y_{i_p}, y_{i_q})}{\left(K(y_{i_p})K(y_{i_q})\right)^{1/2}} \quad \text{for } p \neq q,$$

and we denote by $\rho(\tau_s)$ the least eigenvalue of $M(\tau_s)$. It was first pointed out by Bahri [4] (see also [7] and [9]), that when the self interaction and the interaction between different bubbles are the same size, the function ρ plays a fundamental role in the existence of solutions to problems like (P). Regarding problem (P), Djadli-Malchiodi-Ould Ahmedou [18] observed that such kind of phenomenon appears when $n = 3$.

Now let Z be a pseudogradient of K of Morse-Smale type (that is the intersections of the stable and the unstable manifolds of the critical points of K are transverse).

(1.3)   We assume throughout this paper that $W_s(y_i) \cap W_u(y_j) = \emptyset$ for any $y_i \in \mathcal{F}^+$ and for any $y_j \notin \mathcal{F}^+$, where $W_s(y_i)$ is the stable manifold of y_i and $W_u(y_j)$ is the unstable manifold of y_j for Z.

($\mathbf{H_1}$) Assume that $y_0 \in \mathcal{F}^+$

Let $y_{i_1} \in \mathcal{F}^+ \backslash \{y_0\}$ such that

(1.4) $K(y_{i_1}) = \max\{K(y)/y \in \mathcal{F}^+ \backslash \{y_0\}\}$

and we denote by $k_{i_1} = 4 - i(y_{i_1})$, where $i(y_{i_1})$ is the Morse index of K at y_{i_1}.

($\mathbf{H_2}$) Assume that $i(y_{i_1}) \leq 3$.

We then have the following result :

Theorem 1.2 *Under assumptions (H_1) and (H_2), if the following three conditions hold*

$$(A_0) \quad M(y_0, y_{i_1}) \text{ is nondegenerate}$$
$$(A_1) \quad \rho(y_0, y_{i_1}) < \quad 0$$
$$(A_2) \quad \frac{1}{K(y)} > \frac{1}{K(y_0)} + \frac{1}{K(y_{i_1})} \qquad \forall y \in \mathcal{F}^+ \backslash \{y_0, y_{i_1}\},$$

then problem (1) *has a solution of Morse index k_{i_1} or $k_{i_1} + 1$.*

In contrast to Theorem 1.2, we have the following results based on a topological invariant for some Yamabe type problems introduced by Bahri [5]. To state these results, we need to fix assumptions that we are using and some notation.

($\mathbf{H_3}$) Assume that $\rho(y_0, y_{i_1}) > 0$

Let

$$X = \overline{W_s(y_{i_1})}$$

Under (1.3) and (1.4), we derive that $X = W_s(y_{i_1}) \cup W_s(y_0)$. Thus X is a manifold of dimension k_{i_1} without boundary.

We denote by $C_{y_0}(X)$ the following set

$$C_{y_0}(X) = \{\alpha \delta_{y_0} + (1 - \alpha)\delta_x \, / \, \alpha \in [0, 1], \, x \in X\},$$

where δ_x is the Dirac measure at x.

For λ large enough, we introduce the map $f_\lambda : C_{y_0}(X) \to \Sigma^+$, defined by

$$(\alpha \delta_{y_0} + (1 - \alpha)\delta_x) \longrightarrow \frac{\alpha \delta_{(y_0, \lambda)} + (1 - \alpha)\delta_{(x, \lambda)}}{||\alpha \delta_{(y_0, \lambda)} + (1 - \alpha)\delta_{(x, \lambda)}||},$$

where $||.||$, Σ^+ and $\delta_{(x, \lambda)}$ are defined in the next section by (2.1), (2.2) and (2.3) respectively.

Notice that $C_{y_0}(X)$ and $f_\lambda(C_{y_0}(X))$ are manifolds in dimension $k_{i_1} + 1$, that is, their singularities arise in dimension $k_{i_1} - 1$ and lower, see [5]. We observe that $C_{y_0}(X)$ and

$f_\lambda(C_{y_0}(X))$ are contractible while X is not contractible.

For λ large enough, we also define the intersection number(modulo 2) of $f_\lambda(C_{y_0}(X))$ with $W_s(y_0, y_{i_1})_\infty$

$$\mu(y_0) = f_\lambda(C_{y_0}(X)).W_s(y_0, y_{i_1})_\infty,$$

where $W_s(y_0, y_{i_1})_\infty$ is the stable manifold of the critical point at infinity $(y_0, y_{i_1})_\infty$ (see Corollary 3.2 below) for a decreasing pseudogradient V for the Euler functional associated to (1) which is transverse to $f_\lambda(C_{y_0}(X))$. Thus this number is well defined, see [26].

($\mathbf{H_4}$) Assume that $K(y_0) > 2K(y_{i_1})$.

We then have the following result

Theorem 1.3 *Under assumptions (H_2), (H_3) and (H_4), if $\mu(y_{i_1}) = 0$ then problem (1) has a solution of Morse index k_{i_1} or $k_{i_1} + 1$.*

Now we give a statement more general than Theorem 1.3. To this aim, let $k \geq 1$, and define X as the following

$$X = \overline{\cup_{y \in B_k} W_s(y)}, \quad \text{with } B_k \text{ is any subset in } \{y \in \mathcal{F}^+ / \mathrm{i}(y) = 4 - k\},$$

where $\mathrm{i}(y)$ is the Morse index of K at y.

($\mathbf{H_5}$) We assume that X is a stratified set without boundary (in the topological sense, that is, $X \in \mathcal{S}_k(S_+^4)$, the group of chains of dimension k and $\partial X = 0$).

($\mathbf{H_6}$) Assume that for any critical point z of K in $X \backslash \{y_0\}$, we have $\rho(y_0, z) > 0$.

For $y \in B_k$ we define, for λ large enough, the intersection number(modulo 2)

$$\mu(y) = f_\lambda(C_{y_0}(X)).W_s(y_0, y)_\infty$$

By the above arguments, this number is well defined, see [26].

($\mathbf{H_7}$) Assume that $K(y_0) > 2K(y) \ \forall y \in \mathcal{F}^+ \backslash \{y_0\}$.

Then we have the following theorem

Theorem 1.4 *Under assumptions (H_5), (H_6) and (H_7), if $\mu(y) = 0$ for any $y \in B_k$, then problem (1) has a solution of Morse index k or $k + 1$.*

Next we state a perturbative result for problem (1). To this aim, we set

$$X = \overline{\cup_{y \in \mathcal{F}^+} W_s(y)}$$

($\mathbf{H_8}$) Assume that X is not contractible and denote by m the dimension of the first nontrivial reduced homology group.

We then have

Theorem 1.5 *Assume that assumption (H_8) holds. Thus there exists a constant c_0 independent of K such that if*

$$||K - 1||_{L^\infty(S_+^4)} \leq c_0,$$

then problem (1) has a solution of Morse index $\geq m$.

Lastly, under the assumption (H_8), we can also find the following existence result :

Theorem 1.6 *Under assumption* (H_8), *if the following two conditions hold*

$$(C_1) \quad \text{for any } s, \ M(\tau_s) \text{ is nondegenerate}$$
$$(C_2) \quad \rho(y_i, y_j) < 0 \quad \forall y_i, y_j \in \mathcal{F}^+ \text{ such that } y_i \neq y_j,$$

then problem (1) *has a solution of Morse index* $\geq m$.

The rest of the paper is organized as follows. In section 2, we set up the variational structure and recall some known facts. Lastly, section 3 is devoted to the proofs of our results.

2   Some Known Facts

In this section we recall the functional setting and the variational problem associated to (1). We will also recall some useful previous results.
Problem (1) has a variational structure, the functional being

$$J(u) = \frac{\int_{S^4_+} |\nabla u|^2 + 2\int_{S^4_+} u^2}{\left(\int_{S^4_+} K u^4\right)^{\frac{1}{2}}},$$

defined on the unit sphere of $H^1(S^4_+)$ equipped with the norm

$$||u||^2 = \int_{S^4_+} |\nabla u|^2 + 2\int_{S^4_+} u^2. \tag{2.1}$$

Problem (1) is equivalent to finding the critical points of J subjected to the constraint $u \in \Sigma^+$, where

$$\Sigma^+ = \{u \in \Sigma \,/\, u \geq 0\}, \quad \Sigma = \{u \in H^1(S^4_+)/\ ||u|| = 1\}. \tag{2.2}$$

The Palais-Smale condition fails to be satisfied for J on Σ^+. To characterize the sequences failing the Palais-Smale condition, we need to fix some notation.
For $a \in \overline{S^4_+}$ and $\lambda > 0$, let

$$\delta_{a,\lambda}(x) = \frac{2\sqrt{2}\lambda}{\lambda^2 + 1 + (1 - \lambda^2)\cos d(a, x)}, \tag{2.3}$$

where d is the geodesic distance on $(\overline{S^4_+}, g)$. This function satisfies the following equation

$$-\Delta \delta_{a,\lambda} + 2\delta_{a,\lambda} = \delta^3_{a,\lambda}, \quad \text{in } S^4_+.$$

Let $\varphi_{(a,\lambda)}$ be the function defined on S_+^4 and satisfying

$$\Delta\varphi_{(a,\lambda)} + 2\varphi_{(a,\lambda)} = \Delta\delta_{a,\lambda} + 2\delta_{a,\lambda} \text{ in } S_+^4, \quad \frac{\partial\varphi_{(a,\lambda)}}{\partial\nu} = 0 \text{ on } \partial S_+^4.$$

Now, for $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let us define

$$V(p,\varepsilon) = \{u \in \Sigma / \exists a_1, ..., a_p \in \overline{S_+^4}, \exists\lambda_1, ..., \lambda_p > 0, \exists\alpha_1, ..., \alpha_p > 0 \text{ s.t. } \|u - \sum_{i=1}^p \alpha_i\delta_i\| < \varepsilon,$$

$$|\frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} - 1| < \varepsilon, \lambda_i > \varepsilon^{-1}, \varepsilon_{ij} < \varepsilon \text{ and } \lambda_i d_i < \varepsilon \text{ or } \lambda_i d_i > \varepsilon^{-1}\},$$

where $\delta_i = \delta_{a_i,\lambda_i}$, $d_i = d(a_i, \partial S_+^4)$ and $\varepsilon_{ij}^{-1} = \lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i\lambda_j(1 - \cos d(a_i, a_j))/2$.

The failure of the Palais-Smale condition can be described, following the ideas introduced in [15], [25], [28] as follows:

Proposition 2.1 *Assume that J has no critical point in Σ^+ and let $(u_k) \in \Sigma^+$ be a sequence such that $J(u_k)$ is bounded and $\nabla J(u_k) \to 0$. Then there exist an integer $p \in \mathbb{N}^*$, a sequence $\varepsilon_k > 0$ $(\varepsilon_k \to 0)$ and an extracted subsequence of u_k, again denoted (u_k), such that $u_k \in V(p, \varepsilon_k)$.*

If a function u belongs to $V(p, \varepsilon)$, we assume that, for the sake of simplicity, $\lambda_i d_i < \varepsilon$ for $i \leq q$ and $\lambda_i d_i > \varepsilon^{-1}$ for $i > q$. We consider the following minimization problem for $u \in V(p, \varepsilon)$ with ε small

$$\min\{\|u - \sum_{i=1}^q \alpha_i\delta_{(a_i,\lambda_i)} - \sum_{i=q+1}^p \alpha_i\varphi_{(b_i,\lambda_i)}\|, \alpha_i > 0, \lambda_i > 0, a_i \in \partial S_+^4 \text{ and } b_i \in S_+^4\}. \quad (2.4)$$

We then have the following proposition which defines a parametrization of the set $V(p, \varepsilon)$. It follows from the corresponding statements in [5], [6], [27].

Proposition 2.2 *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon < \varepsilon_p$ and $u \in V(p, \varepsilon)$, the minimization problem (2.4) has a unique solution (up to permutation). In particular, we can write $u \in V(p, \varepsilon)$ as follows*

$$u = \sum_{i=1}^q \bar\alpha_i\delta_{(\bar a_i, \bar\lambda_i)} + \sum_{i=q+1}^p \bar\alpha_i\varphi_{(\bar a_i, \bar\lambda_i)} + v,$$

where $(\bar\alpha_1, ..., \bar\alpha_p, \bar a_1, ..., \bar a_p, \bar\lambda_1, ..., \bar\lambda_p)$ is the solution of (2.4) and $v \in H^1(S_+^n)$ such that

$$(V_0) \quad (v, \psi) = 0 \text{ for } \psi \in \left\{\delta_i, \frac{\partial\delta_i}{\partial\lambda_i}, \frac{\partial\delta_i}{\partial a_i}, \varphi_j, \frac{\partial\varphi_j}{\partial\lambda_j}, \frac{\partial\varphi_j}{\partial a_j} / i \leq q, j > q\right\}.$$

We also have the following proposition whose proof is similar, up to minor modifications to the corresponding statements in [4] (see also [27])

Proposition 2.3 *There exists a C^1 map which, to each $(\alpha_1, ..., \alpha_p, a_1, ..., a_p, \lambda_1, ..., \lambda_p)$ such that $\sum_{i=1}^{p} \alpha_i \delta_i \in V(p, \varepsilon)$ with small ε, associates $\overline{v} = \overline{v}_{(\alpha_i, a_i, \lambda_i)}$ satisfying*

$$J\left(\sum_{i=1}^{q} \alpha_i \delta_i + \sum_{i=q+1}^{p} \alpha_i \varphi_i + \overline{v}\right) = \min\left\{J\left(\sum_{i=1}^{q} \alpha_i \delta_i + \sum_{i=q+1}^{p} \alpha_i \varphi_i + v\right), v \text{ satisfies } (V_0)\right\}.$$

Moreover, there exists $c > 0$ such that the following holds

$$||\overline{v}|| \leq c \left(\sum_{i \leq q} \frac{1}{\lambda_i} + \sum_{i > q} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i > q} \frac{1}{(\lambda_i d_i)^2} + \sum_{k \neq r} \varepsilon_{kr} (log(\varepsilon_{kr}^{-1}))^{1/2}\right).$$

Next we are going to recall a useful expansion of functional J in $V(p, \varepsilon)$.

Proposition 2.4 *[13] For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^{p} \alpha_i \varphi_{(a_i, \lambda_i)} \in V(p, \varepsilon)$, we have the following expansion*

$$J(u) = \frac{S_4^{1/2} \sum \alpha_i^2}{(\sum \alpha_i^4 K(a_i))^{1/2}} \left(1 + \frac{\omega_3}{8S_4} \left(\sum K(a_i)^{-1}\right)^{-1} \left(\sum \left(\frac{-\Delta K(a_i)}{3\lambda_i^2 K(a_i)^2} + \frac{4H(a_i, a_i)}{\lambda_i^2 K(a_i)}\right)\right.\right.$$
$$\left.\left. - \sum_{i \neq j} \frac{2}{(K(a_i)K(a_j))^{1/2}} \left(\varepsilon_{ij} - \frac{2H(a_i, a_j)}{\lambda_i \lambda_j}\right)\right) + o\left(\sum \varepsilon_{kr} + \frac{1}{(\lambda_k d_k)^2}\right)\right),$$

where $S_4 = 64 \int_{\mathbb{R}^4} \frac{dx}{(1+|x|^2)^4}$.

3   Proof of Theorems

Before giving the proof of our theorems, we extract from [13] the characterization of the critical points at infinity of our problem. We recall that the critical points at infinity are the orbits of the gradient flow of J which remain in $V(p, \varepsilon(s))$, where $\varepsilon(s)$, a given function, tends to zero when s tends to $+\infty$ (see [4]).

Proposition 3.1 *[13] Assume that for any s, $M(\tau_s)$ is nondegenerate. Thus, for $p \geq 1$, there exists a pseudogradient W so that the following holds:*
There is a constant $c > 0$ independent of $u = \sum_{i=1}^{q} \alpha_i \delta_i + \sum_{j=q+1}^{p} \alpha_j \varphi_j \in V(p, \varepsilon)$ so that

$$(i) \qquad (-\nabla J(u), W) \geq c\left(\sum_{k \neq r} \varepsilon_{kr} + \sum_{i \leq q} \frac{1}{\lambda_i} + \sum_{j=q+1}^{p} \frac{|\nabla K(a_j)|}{\lambda_j} + \frac{1}{(\lambda_j d_j)^2}\right)$$

(ii) $(-\nabla J(u+\overline{v}), W + \dfrac{\partial \overline{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W)) \geq c\left(\displaystyle\sum_{k\neq r} \varepsilon_{kr} + \sum_{i\leq q}\dfrac{1}{\lambda_i} + \sum_{j=q+1}^{p}\dfrac{|\nabla K(a_j)|}{\lambda_j} + \dfrac{1}{(\lambda_j d_j)^2}\right)$

(iii) $|W|$ *is bounded. Furthermore, the only case where the maximum of the λ_i's is not bounded is when each point a_i is close to a critical point y_{j_i} of K with $j_i \neq j_k$ for $i \neq k$ and $\rho(y_{i_1}, ..., y_{i_p}) > 0$, where $\rho(y_{i_1}, ..., y_{i_p})$ denotes the least eigenvalue of $M(y_{i_1}, ..., y_{i_p})$.*

Corollary 3.2 *[13] Assume that for any s $M(\tau_s)$ is nondegenerate, and assume further that J has no critical point in Σ^+. Then the only critical points at infinity of J correspond to*

$$\sum_{j=1}^{p} K(y_{i_j})^{-1/2}\varphi_{(y_{i_j},\infty)}, \qquad with \ p \in \mathbb{N}^* \ and \ \rho(y_{i_1}, ..., y_{i_p}) > 0.$$

In addition, in the neighborhood of such a critical point at infinity, we can find a change of variable

$$(a_1, ..., a_p, \lambda_1, ..., \lambda_p) \to (\tilde{a}_1, ..., \tilde{a}_p, \tilde{\lambda}_1, ..., \tilde{\lambda}_p) := (\tilde{a}, \tilde{\lambda})$$

such that

$$J\left(\sum_{i=1}^{p} \alpha_i \varphi_i + \overline{v}\right) = \psi(\alpha, \tilde{a}, \tilde{\lambda}) := \dfrac{8 S_4^{1/2}\sum_{i=1}^{p}\alpha_i^2}{(\sum_{i=1}^{p}\alpha_i^4 K(a_i))^{1/2}}\left(1 + (c-\eta)\left(\sum_{i=1}^{p}\dfrac{1}{K(y_{j_i})}\right)^{-1} {}^t\Lambda M(\tau_p)\Lambda\right)$$

where $\alpha = (\alpha_1, ..., \alpha_p)$, c is a positive constant, η is a small positive constant, ${}^t\Lambda = (\tilde{\lambda}_1, ..., \tilde{\lambda}_p)$, $\tau_p = (y_{j_i}, ..., y_{j_p})$.

Now we are ready to prove our theorems.

Proof of Theorem 1.1 For $\eta > 0$ small enough, we introduce, following [13], this neighborhood of Σ^+

$$V_\eta(\Sigma^+) = \{u \in \Sigma / \ e^{2J(u)}J(u)^3|u^-|_{L^4}^2 < \eta\},$$

where $u^- = \max(0, -u)$.

Recall that, from Proposition 3.1 we have a vector field W defined in $V(p, \varepsilon)$ for $p \geq 1$. Outside $\cup_{p\geq 1}V(p, \varepsilon/2)$, we will use $-\nabla J$ and our global vector field Z will be built using a convex combination of W and $-\nabla J$. $V_\eta(\Sigma^+)$ is invariant under the flow line generated by Z (see [9]). Arguing by contradiction, we assume that J has no critical point in $V_\eta(\Sigma^+)$. For any y critical point of K, set

$$c_\infty(y) = \left(\dfrac{S_4}{K(y)}\right)^{1/2}.$$

Since y_0 is the unique absolute maximum of K, we derive that

$$c_\infty(y_0) < c_\infty(y), \qquad \forall y \neq y_0,$$

where y is any critical point of K.

Let $u_0 \in \Sigma^+$ such that

$$c_\infty(y_0) < J(u_0) < \inf_{y/y \neq y_0, \nabla K(y)=0} c_\infty(y) \tag{3.1}$$

and let $\eta(s, u_0)$ be the one parameter group generated by Z. It is known that $|\nabla J|$ is lower bounded outside $V(p, \varepsilon/2)$, for any $p \in \mathbb{N}^*$ and for ε small enough, by a fixed constant which depends only on ε. Thus the flow line $\eta(s, u_0)$ cannot remain outside of the set $V(p, \varepsilon/2)$. Furthermore, if the flow line travels from $V(p, \varepsilon/2)$ to the boundary of $V(p, \varepsilon)$, $J(\eta(s, u_0))$ will decrease by a fixed constant which depends on ε. Then, this travel cannot be repeated in an infinite time. Thus there exist p_0 and s_0 such that the flow line enters into $V(p_0, \varepsilon/2)$ and it does not exit from $V(p_0, \varepsilon)$. Since u_0 satisfies (3.1), we derive that $p_0 = 1$, thus, for $s \geq s_0$,

$$\eta(s, u_0) = \alpha_1 \varphi_{(x_1(s), \lambda_1(s))} + v(s)$$

Using again (3.1), we deduce that $x_1(s)$ is outside $\mathcal{V}(y, \tau)$ for any $y \in \mathcal{F}^+ \backslash \{y_0\}$, where $\mathcal{V}(y, \tau)$ is a neighborhood of y and where τ is a small positive real. Now, by assumptions of Theorem 1.1 and by the construction of a pseudogradient Z, we derive that $\lambda_1(s)$ remains bounded along the flow lines of Z. Thus we obtain

$$|\nabla J(\eta(s, u_0)).Z(\eta(s, u_0))| \geq c > 0 \quad \forall s \geq 0,$$

where c depends only on u_0.

Then when s goes to $+\infty$, $J(\eta(s, u_0))$ goes to $-\infty$ and this yields a contradiction. Thus there exists a critical point of J in $V_\eta(\Sigma^+)$. Arguing as in [9], we prove that such a critical point is positive and hence our result follows. $\square$

Now before giving the proof of Theorem 1.2, we state the following lemma. Its proof is very similar to the proof of Corollary B.3 of [6](see also [5]), so we will omit it.

Lemma 3.3 *Let a_1, $a_2 \in S_+^4$, α_1, $\alpha_2 > 0$ and λ large enough. For $u = \alpha_1 \varphi_{(a_1, \lambda)} + \alpha_2 \varphi_{(a_2, \lambda)}$, we have*

$$J\left(\frac{u}{||u||}\right) \leq S_4^{1/2} \left(\frac{1}{K(a_1)} + \frac{1}{K(a_2)}\right)^{1/2} (1 + o(1)).$$

Proof of Theorem 1.2   Again, we argue by contradiction. We assume that J has no critical point in $V_\eta(\Sigma^+)$. Let

$$c_\infty(y_0, y_{i_1}) = S_4^{1/2} \left(\frac{1}{K(y_0)} + \frac{1}{K(y_{i_1})}\right)^{1/2}$$

We observe that under the assumption (A_1) of Theorem 1.2, (y_0, y_{i_1}) is not a critical point at infinity of J. Using Corollary 3.2 and the assumption (A_2) of Theorem 1.2, it follows

that the only critical points at infinity of J under the level $c_1 = c_\infty(y_0, y_{i_1}) + \varepsilon$, for ε small enough, are $\varphi_{(y_0, \infty)}$ and $\varphi_{(y_{i_1}, \infty)}$. The unstable manifolds at infinity of such critical points at infinity, $W_u(y_0)_\infty$, $W_u(y_{i_1})_\infty$ can be described, using Corollary 3.2, as the product of $W_s(y_0)$, $W_s(y_{i_1})$ (for a pseudogradient of K) by $[A, +\infty[$ domain of the variable λ, for some positive number A large enough.

Since J has no critical point, it follows that $J_{c_1} = \{u \in \Sigma^+ / J(u) \leq c_1\}$ retracts by deformation on $X_\infty = W_u(y_0)_\infty \cup W_u(y_{i_1})_\infty$ (see Sections 7 and 8 of [8]) which can be parametrized by $X \times [A, +\infty[$, where $X = \overline{W_s(y_{i_1})}$.

Under (1.3) and (1.4) (see the first section), $X = W_s(y_0) \cup W_s(y_{i_1})$. Thus X is a manifold in dimension k_{i_1} without boundary.

We claim now that X_∞ is contractible in J_{c_1}. Indeed, let $h : [0, 1] \times X \times [A, +\infty[\longmapsto \Sigma^+$ defined by

$$(t, x, \lambda) \longmapsto \frac{t\varphi_{(y_0, \lambda)} + (1 - t)\varphi_{(x, \lambda)}}{||t\varphi_{(y_0, \lambda)} + (1 - t)\varphi_{(x, \lambda)}||}$$

h is continuous and satisfies

$$h(0, x, \lambda) = \frac{\varphi_{(x, \lambda)}}{||\varphi_{(x, \lambda)}||} \quad \text{and} \quad h(1, x, \lambda) = \frac{\varphi_{(y_0, \lambda)}}{||\varphi_{(y_0, \lambda)}||}.$$

In addition, since $K(x) \geq K(y_{i_1})$ for any $x \in X$, it follows from Lemma 3.3 that $J(h(t, x, \lambda)) < c_1$, for each $(t, x, \lambda) \in [0, 1] \times X \times [A, +\infty[$. Thus the contraction h is performed under the level c_1. We derive that X_∞ is contractible in J_{c_1}, which retracts by deformation on X_∞, therefore X_∞ is contractible leading to the contractibility of X, which is a contradiction, since X is a manifold in dimension k_{i_1} without boundary. Hence there exists a critical point of J in $V_\eta(\Sigma^+)$. Arguing as in [9], we prove that such a critical point is positive. Now we are going to show that such a critical point has a Morse index equal to k_{i_1} or $k_{i_1} + 1$.

Using a dimension argument and since $h([0, 1], X_\infty)$ is a manifold in dimension $k_{i_1} + 1$, we derive that the Morse index of such a critical point is $\leq k_{i_1} + 1$.

Now, arguing by contradiction, we assume that the Morse index is $\leq k_{i_1} - 1$. Perturbing, if necessary J, we may assume that all the critical points of J are nondegenerate and have their Morse index $\leq k_{i_1} - 1$.

Such critical points do not change the homological group in dimension k_{i_1} of level sets of J.

Now let $c_\infty(y_{i_1}) = S_4^{1/2} K(y_{i_1})^{-1/2}$ and let ε be a small positive real. Since X_∞ defines a homological class in dimension k_{i_1} which is trivial in J_{c_1}, but not trivial in $J_{c_\infty(y_{i_1}) + \varepsilon}$, our result follows. $\qquad \square$

Proof of Theorem 1.3 We notice that the assumption (H_3) implies that (y_0, y_{i_1}) is a critical point at infinity of J. Now, arguing by contradiction, we assume that (1) has no solution. We claim that $f_\lambda(C_{y_o}(X))$ retracts by deformation on $X \cup W_u(y_0, y_{i_1})_\infty$. Indeed, let

$$u = \alpha\varphi_{(y_0, \lambda)} + (1 - \alpha)\varphi_{(x, \lambda)} \in f_\lambda(C_{y_0}(X)),$$

the action of the flow of the pseudogradient Z defined in the proof of Theorem 1.1 is essentially on α (see [5] and [11]).

- If $\alpha < 1/2$, the flow of Z brings α to zero and thus u goes in this case to $\overline{W_u(y_0)}_\infty \equiv \{y_0\}$.

- If $\alpha > 1/2$, the flow of Z brings α to 1 and thus u goes, in this case, to $\overline{W_u(y_{i_1})}_\infty \equiv X_\infty$.

- If $\alpha = 1/2$, since only x can move then y_0 remains one of the points of concentration of u and x goes to $W_s(y_i)$, where $y_i = y_{i_1}$ or $y_i = y_0$ and two cases may occur :

- In the first case, that is, $y_i = y_{i_1}$, u goes to $W_u(y_0, y_{i_1})_\infty$.

- In the second case, that is, $y_i = y_0$, there exists $s_0 \geq 0$ such that $x(s_0)$ is close to y_0. Thus, using Lemma 3.3, we have the following inequality

$$J(u(s_0)) \leq c_\infty(y_0, y_0) + \gamma := c_2,$$

where $c_\infty(y_0, y_0) = S_4^{1/2}(2/K(y_0))^{1/2}$ and where γ is a positive constant small enough.

Now, using assumption (H_4), it follows from Corollary 3.2 that J_{c_2} retracts by deformation on $W_u(y_0)_\infty \equiv \{y_0\}$ and thus u goes to $W_u(y_0)_\infty$. Therefore $f_\lambda(C_{y_0}(X))$ retracts by deformation on $X_\infty \cup W_u(y_0, y_{i_1})_\infty$.

Now, since $\mu(y_{i_1}) = 0$, it follows that this strong retract does not intersect $W_u(y_0, y_{i_1})_\infty$ and thus it is contained in X_∞. Therefore X_∞ is contractible, leading to the contractibility of X, which is a contradiction, since X is a manifold of dimension k_{i_1} without boundary. Hence (1) admits a solution. Now, using the same arguments as those used in the proof of Theorem 1.2, we easily derive that the Morse index of the solution provided above is equal to k_{i_1} or $k_{i_1} + 1$. Thus our result follows.   □

Proof of Theorem 1.4   Assume that (1) has no solution. Using the same arguments as those in the proof of Theorem 1.3, we deduce that $f_\lambda(C_{y_0}(X))$ retracts by deformation on

$$X_\infty \cup (\cup_{y \in B_k} W_u(y_0, y)_\infty) \cup D,$$

where $D \subset \sigma$ is a stratified set and where $\sigma = \cup_{y \in X \setminus B_k} W_u(y_0, y)_\infty$ is a manifold in dimension at most k.

Since $\mu(y) = 0$ for each $y \in B_k$, $f_\lambda(C_{y_0}(X))$ retracts by deformation on $X_\infty \cup D$, and therefore $H_*(X_\infty \cup D) = 0$, for all $* \in \mathbb{N}^*$, since $f_\lambda(C_{y_0}(X))$ is a contractible set. Using the exact homology sequence of $(X_\infty \cup D, X_\infty)$, we obtain

$$... \to H_{k+1}(X_\infty \cup D) \to^\pi H_{k+1}(X_\infty \cup D, X_\infty) \to^\partial H_k(X_\infty) \to^i H_k(X_\infty \cup D) \to ...$$

Since $H_*(X_\infty \cup D) = 0$, for all $* \in \mathbb{N}^*$, then $H_k(X_\infty) = H_{k+1}(X_\infty \cup D, X_\infty)$. In addition, $(X_\infty \cup D, X_\infty)$ is a stratified set of dimension at most k, then $H_{k+1}(X_\infty \cup D, X_\infty) = 0$, and therefore $H_k(X_\infty) = 0$. This implies that $H_k(X) = 0$ (recall that $X_\infty \equiv X \times [A, \infty)$). This yields a contradiction since X is a manifold in dimension k without boundary. Then, arguing as in the end of the proof of Theorem 1.3, our theorem follows.   □

Proof of Theorem 1.5   We argue by contradiction. Assume that (1) has no solution.

Let $c_1 = \frac{3}{2} S_4^{1/2}$. Using the expansion of J, see Proposition 2.4, we derive that there exists a constant c_0 independent of K such that if $||K-1||_{L^\infty(S_+^4)} \leq c_0$, then the following holds

$$J(u) < c_1, \quad \forall u \in V(1,\varepsilon) \quad \text{and} \quad J(u) > c_1 \quad \forall u \in V(p,\varepsilon) \text{ with } p \geq 2,$$

where ε is a small positive real.

Therefore, it follows from Corollary 3.2 that the critical points at infinity of J under the level c_1 are in one to one correspondence with the critical points y of K such that $y \in \mathcal{F}^+$. Since J has no critical points in Σ^+, it follows that J_{c_1} retracts by deformation on $X_\infty = \cup_{y_i \in \mathcal{F}^+} W_u(y_i)_\infty$ (see sections 7 and 8 of [8]) which can be parametrized by $X \times [A, +\infty[$ (we recall that $X = \cup_{y \in \mathcal{F}^+} \overline{W_s(y)}$).

Now we claim that X_∞ is contractible in J_{c_1}. Indeed, since S_+^4 is a contractible set, we deduce that there exists a contraction $h : [0,1] \times X \to S_+^4$, h continuous and satisfies for any $a \in X$, $h(0,a) = a$ and $h(1,a) = a_0$ a point of S_+^4. Such a contraction gives rise to the following contraction $\tilde{h} : [0,1] \times X_\infty \to \Sigma^+$ defined by

$$[0,1] \times X \times [A, +\infty[\ni (t,a,\lambda) \longmapsto \varphi_{(h(t,a),\lambda)} + \bar{v} \in \Sigma^+.$$

For $t = 0$, $\varphi_{(h(0,a),\lambda)} + \bar{v} = \varphi_{(a,\lambda)} + \bar{v} \in X_\infty$. $\tilde{h}$ is continuous and $\tilde{h}(1,a,\lambda) = \varphi_{(a_0,\lambda)} + \bar{v}$, hence our claim follows.

Now, using Proposition 2.4, we deduce that

$$J(\varphi_{(h(t,a),\lambda)} + \bar{v}) \sim S_4^{\frac{1}{2}}(K(h(t,a)))^{\frac{-1}{2}} \left(1 + O(A^{-2})\right).$$

Choosing c_0 small enough and A large enough, we then have $J(\varphi_{(h(t,a),\lambda)}) < c_1$. Therefore such a contraction is performed under the level c_1, so X_∞ is contractible in J_{c_1}, which retracts by deformation on X_∞, therefore X_∞ is contractible leading to the contractibility of X, which is in contradiction with assumption (H_8). We derive that (1) has a solution. Lastly, the same arguments as those in the proof of Theorem 1.2 easily give the desired estimate for the Morse index of the solution provided above and therefore our result follows. $\square$

Proof of Theorem 1.6   Arguing by contradiction, we assume that (1) has no solution. Notice that under the assumption of our theorem, J has no critical point at infinity having more or equal to two masses in its description and therefore the only critical points at infinity of J are $\varphi(y,\infty)$ such that $y \in \mathcal{F}^+$. Since J has no critical point, it follows that Σ^+ retracts by deformation on $X_\infty = \cup_{y \in \mathcal{F}^+} W_u(y)_\infty$, see [8]. Thus we conclude that X_∞ is contractible since Σ^+ is a contractible set. So X is contractible and this is a contradiction with our assumption. Hence (1) has a solution and as above, we derive that the Morse index of such a solution is $\geq m$, therefore our result follows. $\square$

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